NumBabies HW01

Q1. Finding Inverses

a) Suppose we want to find the multiplicative inverse of a real number a. The answer is, of course, $1/a$. A numerical way to state this is to solve for x such that

$$f(x) = \frac{1}{x} - a = 0.$$

This becomes a root-finding problem. Show that Newton's method gives

$$x_{n+1} = x_n \left(2 - a x_n\right).$$

Plug in numbers to verify that this is correct.

b) Show that the same formula also works for finding the inverse of a matrix A:

$$X_{n+1} = X_n \left(2I - A X_n\right).$$

Demonstrate this with a 2×2 matrix. (Not all initial guesses will work.)

Q2. Complex i as a 2×2 Matrix

If we represent the complex number i as

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

in Julia you can write:

```julia
imat = [0 -1; 1 0]
```

Show how

$$\begin{aligned} i^2 &= -1 \\ e^{i\pi} &= -1 \\ e^{i\pi/2} &= i \end{aligned}$$

are realized in this matrix representation.

Q3. Midpoint Versus Simpson's Rule

a) Write Julia functions `midpoint(f, a, b, N)` and `simpson(f, a, b, N)` that implement these integration rules to compute

$$\int_a^b f(x)\, dx$$

for a general function f using N grid points.

b) Approximate the following integrals (find a suitable mapping function if needed):

Function	Exact (for reference)
$\int_0^\infty e^{-x^2}\, dx$	$\sqrt{\pi}/2$
$\int_{-\infty}^\infty e^{-x^4}\, dx$	$\frac{1}{2}\Gamma(1/4)$
$\int_0^\infty \sin(x^2)\, dx$	$\sqrt{\pi/8}$

c) Tabulate the numerical results for various values of N: $5, 10, 20, 50, 100, 1000$. Compare the numerical values with the exact results.

Q4. Roots via Companion Matrix

For a polynomial of degree n,

$$p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$

the **companion matrix** C is

$$C = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix}_{n\times n}.$$

The **eigenvalues of** C are the roots of $p(x)$.

a) Write a Julia function `companion_roots(coeffs)` that takes a vector of the c_j values as input and computes all roots.

b) Test your function on the polynomials

$$x^4 - x^3 - x^2 - x - 1 = 0$$

and

$$x^5 - 1 = 0.$$

Report all roots.

c) Visualize these roots by plotting them on the complex plane.

Q5. Generalized Gaussian Integrals

a) The Gaussian integral

$$\int_{-\infty}^{\infty} e^{-Ax^2}\, dx = \sqrt{\frac{\pi}{A}}$$

generalizes to

$$\int dx_1\, dx_2 \cdots dx_N\, e^{-\vec{x}^{\,t} A \vec{x}} = \sqrt{\frac{\pi^N}{\det A}}.$$

Here A should be an $N \times N$ positive-definite matrix, and

$$\vec{x}^{\,t} A \vec{x} = \sum_{i,j} x_i x_j A_{ij}.$$

Write a program to verify the result for $N = 2, 3, 6$. Start with an A that is positive definite. (Please avoid diagonal matrices.)

b) Can you perform calculations for $N = 10$ with your method of choice?